Divisibility properties of sporadic Apéry-like numbers

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Abstract

In 1982, Gessel showed that the Apéry numbers associated to the irrationality of $\zeta(3)$ satisfy Lucas congruences. Our main result is to prove corresponding congruences for all sporadic Apéry-like sequences. In several cases, we are able to employ approaches due to McIntosh, Samol–van Straten and Rowland–Yassawi to establish these congruences. However, for the sequences often labeled s_{18} and (η) we require a finer analysis.

As an application, we investigate modulo which numbers these sequences are periodic. In particular, we show that the Almkvist–Zudilin numbers are periodic modulo 8, a special property which they share with the Apéry numbers. We also investigate primes which do not divide any term of a given Apéry-like sequence.

1 Introduction

In his surprising proof [Apé79], [Poo79] of the irrationality of $\zeta(3)$, R. Apéry introduced the sequence

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{1}$$

which has since been referred to as the Apéry sequence. It was shown by I. Gessel [Ges82, Theorem 1] that, for any prime p, these numbers satisfy the *Lucas congruences*

$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p}, \tag{2}$$

where $n = n_0 + n_1 p + \cdots + n_r p^r$ is the expansion of n in base p. Initial work of F. Beukers [Beu02] and D. Zagier [Zag09], which was extended by G. Almkvist,

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W. Zudilin [AZ06] and S. Cooper [Coo12], has complemented the Apéry numbers with a, conjecturally finite, set of sequences, known as Apéry-like, which share (or are believed to share) many of the remarkable properties of the Apéry numbers, such as connections to modular forms [SB85], [Beu87], [AO00] or supercongruences [Beu85], [Cos88], [CCS10], [OS11], [OS13], [OSS14]. After briefly reviewing Apéry-like sequences in Section 2, we prove in Sections 3 and 4 our main result that all of these sequences also satisfy the Lucas congruences (2). For all but two of the sequences, we establish these congruences in Section 3 by extending a general approach provided by R. McIntosh [McI92]. The main difficulty, however, lies in establishing these congruences for the sequence (η). For this sequence, and to a lesser extent for the sequence s_{18}, we require a much finer analysis, which is given separately in Section 4.

In the approaches of Gessel and McIntosh, binomial sums, like (1), are used to derive Lucas congruences. Other known approaches to proving Lucas congruences for a sequence $C(n)$ are based on expressing $C(n)$ as the constant terms of powers of a Laurent polynomial or as the diagonal coefficients of a multivariate algebraic function. However, neither of these approaches is known to apply, for instance, to the sequence (η). In the first approach, one seeks a Laurent polynomial $\Lambda(\boldsymbol{x}) = \Lambda(x_1, \ldots, x_d)$ such that $C(n)$ is the constant term of $\Lambda(\boldsymbol{x})$. In that case, we write $C(n) = \operatorname{ct} \Lambda(\boldsymbol{x})^n$ for brevity. If the Newton polyhedron of $\Lambda(\boldsymbol{x})$ has the origin as its only interior integral point, the results of K. Samol and D. van Straten [SvS09] (see also [MV13]) apply to show that $C(n)$ satisfies the *Dwork congruences*

$$C(p^r m + n)C(\lfloor n/p \rfloor) \equiv C(p^{r-1}m + \lfloor n/p \rfloor)C(n) \pmod{p^r} \tag{3}$$

for all primes p and all integers $m, n \geqslant 0$, $r \geqslant 1$. The case $r = 1$ of these congruences is equivalent to the Lucas congruences (2) for the sequence $C(n)$. For instance, in the case of the Apéry numbers (1), we have [Str14, Remark 1.4]

$$A(n) = \operatorname{ct} \left[\frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz} \right]^n,$$

from which one may conclude that the Apéry numbers satisfy the congruences (3), generalizing (2). Similarly, for the sequence (η), one may derive from the binomial sum (22), using G. Egorychev's method of coefficients [Ego84], that its nth term is given by $\operatorname{ct} \Lambda(x, y, z)^n$, where

$$\Lambda(x, y, z) = \left(1 - \frac{1}{xy(1+z)^5}\right) \frac{(1+x)(1+y)(1+z)^4}{z^3}.$$

However, $\Lambda(x, y, z)$ is not a Laurent polynomial, and it is unclear if and how one could express the sequence (η) as constant terms of powers of an appropriate Laurent polynomial. As a second general approach, E. Rowland and R. Yassawi [RY13] show that Lucas congruences hold for a certain class of sequences that can be represented as the diagonal Taylor coefficients of $1/Q(\boldsymbol{x})^{1/s}$, where $s \geqslant 1$ is an integer and $Q(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ is a multivariate polynomial. Again, while

such representations are known for some Apéry-like sequences, see, for instance, [Str14], no suitable representations are available for the sequences (η) or s_{18}.

It was conjectured by S. Chowla, J. Cowles and M. Cowles [CCC80] and subsequently proven by I. Gessel [Ges82] that

$$A(n) \equiv \begin{cases} 1, & \text{if } n \text{ is even,} \\ 5, & \text{if } n \text{ is odd,} \end{cases} \pmod 8. \tag{4}$$

The congruences (4) show that the Apéry numbers are periodic modulo 8, and it was recently demonstrated by E. Rowland and R. Yassawi [RY13] that they are not eventually periodic modulo 16, thus answering a question of Gessel. The Apéry numbers are also periodic modulo 3 (see (47)) and their values modulo 9 are characterized by an extension of the Lucas congruences [Ges82]; see also the recent generalizations [KM15] of C. Krattenthaler and T. Müller, who characterize generalized Apéry numbers modulo 9. As an application of the Lucas congruences established in Sections 3 and 4, we address in Section 5 the natural question to which extent results like (4) are true for Apéry-like numbers in general. In particular, we show in Theorem 5.3 that the Almkvist–Zudilin numbers are periodic modulo 8 as well.

The primes $2, 3, 7, 13, 23, 29, 43, 47, \ldots$ do not divide any Apéry number $A(n)$, and E. Rowland and R. Yassawi [RY13] pose the question whether there are infinitely many such primes. While this question remains open, we offer numerical and heuristic evidence that a positive proportion of the primes, namely, about $e^{-1/2} \sim 0.6065$, do not divide any Apéry number. In Section 6, we investigate the analogous question for other Apéry-like numbers, and prove that Cooper's sporadic sequences [Coo12] behave markedly differently. Indeed, for any given prime p, a fixed proportion of the last of the first p terms of these sequences is divisible by p. In the case of sums of powers of binomial coefficients, such a result has been proven by N. Calkin [Cal98].

2 Review of Apéry-like numbers

Along with the Apéry numbers $A(n)$, defined in (1), R. Apéry also introduced the sequence

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

which allowed him to (re)prove the irrationality of $\zeta(2)$. This sequence is the solution of the three-term recursion

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \tag{5}$$

with the choice of parameters $(a, b, c) = (11, 3, -1)$ and initial conditions $u_{-1} = 0$, $u_0 = 1$. Because we divide by $(n+1)^2$ at each step, it is not to be expected that the recursion (5) should have an integer solution. Inspired by F. Beukers [Beu02], D. Zagier [Zag09] conducted a systematic search for other choices of

the parameters (a, b, c) for which the solution to (5), with initial conditions $u_{-1} = 0$, $u_0 = 1$, is integral. After normalizing, and apart from degenerate cases, he discovered four hypergeometric, four Legendrian as well as six sporadic solutions. It is still open whether further solutions exist or even that there should be only finitely many solutions. The six sporadic solutions are reproduced in Table 1. Note that each binomial sum included in this table certifies that the corresponding sequence indeed consists of integers.

(a, b, c)	[Zag09]	[AvSZ11]	$A(n)$
$(7, 2, -8)$	**A**	(a)	$\sum_k \binom{n}{k}^3$
$(11, 3, -1)$	**D**	(b)	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}$
$(10, 3, 9)$	**C**	(c)	$\sum_k \binom{n}{k}^2 \binom{2k}{k}$
$(12, 4, 32)$	**E**	(d)	$\sum_k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$
$(9, 3, 27)$	**B**	(f)	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$
$(17, 6, 72)$	**F**	(g)	$\sum_{k,l} (-1)^k 8^{n-k} \binom{n}{k} \binom{k}{l}^3$

Table 1: The six sporadic solutions of (5)

Similarly, the Apéry numbers $A(n)$, defined in (1), are the solution of the three-term recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}, \tag{6}$$

with the choice of parameters $(a, b, c, d) = (17, 5, 1, 0)$ and initial conditions $u_{-1} = 0$, $u_0 = 1$. Systematic computer searches for further integer solutions have been performed by G. Almkvist and W. Zudilin [AZ06] in the case $d = 0$ and, more recently, by S. Cooper [Coo12], who introduced the additional parameter d. As in the case of (5), apart from degenerate cases, only finitely many sequences have been discovered. In the case $d = 0$, there are again six sporadic sequences, which are recorded in Table 2. Moreover, by general principles (see [Coo12, Eq. (17)]), each of the sequences in Table 1 times $\binom{2n}{n}$ is an integer solution of (6) with $d \neq 0$. Apart from such expected solutions, Cooper also found three additional sporadic solutions, including

$$s_{18}(n) = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right], \tag{7}$$

for $n \geqslant 1$, with $s_{18}(0) = 1$, as well as s_7 and s_{10}, which are included in Table 2. Remarkably, these sequences are again connected to modular forms [Coo12] (the subscript refers to the level) and satisfy supercongruences, which are proved in [OSS14]. Indeed, it was the corresponding modular forms and Ramanujan-type series for $1/\pi$ that led Cooper to study these sequences, and the binomial expressions for s_7 and s_{18} were found subsequently by Zudilin (sequence s_{10} was well-known before).

(a,b,c,d)	[AvSZ11]	$A(n)$
$(7,3,81,0)$	(δ)	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125,0)$	(η)	$\sum_{k=0}^{\lfloor n/5 \rfloor} (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right)$
$(10,4,64,0)$	(α)	$\sum_k \binom{n}{k}^2 \binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16,0)$	(ϵ)	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$
$(9,3,-27,0)$	(ζ)	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1,0)$	(γ)	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$
$(13,4,-27,3)$	s_7	$\sum_k \binom{n}{k}^2 \binom{n+k}{k}\binom{2k}{n}$
$(6,2,-64,4)$	s_{10}	$\sum_k \binom{n}{k}^4$
$(14,6,192,-12)$	s_{18}	defined in (7)

Table 2: The sporadic solutions of (6)

3 Lucas congruences

It is a well-known and beautiful classical result of Lucas [Luc78] that the binomial coefficients satisfy the congruences

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r} \pmod{p}, \tag{8}$$

where p is a prime and n_i, respectively k_i, are the p-adic digits of n and k. That is, $n = n_0 + n_1 p + \cdots + n_r p^r$ and $k = k_0 + k_1 p + \cdots + k_r p^r$ are the expansions of n and k in base p. Correspondingly, a sequence $a(n)$ is said to satisfy *Lucas*

congruences, if the congruences

$$a(n) \equiv a(n_0)a(n_1)\cdots a(n_r) \pmod{p} \tag{9}$$

hold for all primes p. It was shown by I. Gessel [Ges82, Theorem 1] that the Apéry numbers $A(n)$, defined in (1), satisfy Lucas congruences. E. Deutsch and B. Sagan [DS06, Theorem 5.9] show that the Lucas congruences (9) in fact hold for the family of generalized Apéry sequences

$$A_{r,s}(n) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s, \tag{10}$$

with r and s positive integers. This family includes the sequences (a), (b) from Table 1, and the sequences (γ), s_{10} from Table 2. The purpose of this section and Section 4 is to show that, in fact, all the Apéry-like sequences in Tables 1 and 2 satisfy the Lucas congruences (9). Using and extending the general framework provided by R. McIntosh [McI92, Theorem 6], which we review below, we are able to prove this claim for all of the sequences in the two tables, with the exception of the two sequences (η) and s_{18}, for which we require a much finer analysis, which is given in Section 4.

Theorem 3.1. *Each of the sequences from Tables 1 and 2 satisfies the Lucas congruences* (9)*.*

Remark 3.2. The Lucas congruences (9), in general, do not extend to prime powers. However, it is shown in [Ges82], and generalized in [KM15], that the Lucas congruences modulo 3 for the Apéry numbers extend to hold modulo 9.

On the other hand, numerical evidence suggests that all the Apéry-like sequences from Tables 1 and 2 in fact satisfy the Dwork congruences (3). While Theorem 3.1 proves the case $r = 1$ of these congruences, it would be desirable to establish the corresponding congruences modulo higher powers of primes.

Following [McI92], we say that a function $L : \mathbb{Z}_{\geqslant 0}^2 \to \mathbb{Z}$ has the *double Lucas property* (**DLP**) if $L(n, k) = 0$, for $k > n$, and if

$$L(n, k) \equiv L(n_0, k_0)L(n_1, k_1)\cdots L(n_r, k_r) \pmod{p}, \tag{11}$$

for every prime p. Here, as in (8), n_i and k_i are the p-adic digits of n and k, respectively. Equation (8) shows that the binomial coefficients $\binom{n}{k}$ are a **DLP** function. More generally, it is shown in [McI92, Theorem 6] that, for positive integers $r_0, r_1, \ldots, r_m$,

$$L(n, k) = \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \binom{n+2k}{k}^{r_2} \cdots \binom{n+mk}{k}^{r_m} \tag{12}$$

is a **DLP** function. For instance, choosing the exponents as $r_i = 1$, we find that the multinomial coefficient

$$\binom{n+mk}{k, k, \ldots, k, n-k} = \frac{(n+mk)!}{k!^{m+1}(n-k)!}$$

is a **DLP** function for any integer $m \geqslant 0$.

Suppose that $L(n, k)$ is a **DLP** function and that $G(n)$ and $H(n)$ are **LP** functions, that is, the sequences $G(n)$ and $H(n)$ satisfy the Lucas congruences (9). Then, as shown in [McI92, Theorem 5],

$$F(n) = \sum_{k=0}^{n} L(n, k) G(k) H(n-k) \tag{13}$$

is an **LP** function. Note that (12) and (13) combined are already sufficient to prove that the generalized Apéry sequences, defined in (10), satisfy Lucas congruences. In order to apply this machinery more generally, and prove Theorem 3.1, our next results extend the repertoire of **DLP** functions. In fact, it turns out that we need a natural extension of the Lucas property to the case of three variables. We say that a function $M : \mathbb{Z}_{\geqslant 0}^3 \to \mathbb{Z}$ has the *triple Lucas property* (**TLP**) if $M(n, k, j) = 0$, for $j > n$, and if

$$M(n, k, j) \equiv M(n_0, k_0, j_0) \cdots M(n_r, k_r, j_r) \pmod{p}, \tag{14}$$

for every prime p, where n_i, k_i and j_i are the p-adic digits of n, k and j, respectively. It is straightforward to prove the following analog of (13) for **TLP** functions.

Lemma 3.3. *If $M(n, k, j)$ is a **TLP** function, then*

$$L(n, k) = \sum_{j=0}^{n} M(n, k, j)$$

*satisfies the double Lucas congruences (11). In particular, if $L(n, k) = 0$, for $k > n$, then $L(n, k)$ is a **DLP** function.*

Proof. Let p be a prime. It is enough to show that, for any nonnegative integers n_0, n', k_0, k' such that $n_0 < p$ and $k_0 < p$,

$$L(n_0 + n'p, k_0 + k'p) \equiv L(n_0, k_0) L(n', k') \pmod{p}. \tag{15}$$

Since the sum defining $L(n, k)$ is naturally supported on $j \in \{0, 1, \ldots, n\}$, we may extend it over all $j \in \mathbb{Z}$. Modulo p, we have

$$\begin{aligned} L(n, k) &= \sum_{j \in \mathbb{Z}} M(n, k, j) \\ &= \sum_{j_0=0}^{p-1} \sum_{j' \in \mathbb{Z}} M(n, k, j_0 + j'p) \\ &\equiv \sum_{j_0 \in \mathbb{Z}} \sum_{j' \in \mathbb{Z}} M(n_0, k_0, j_0) M(n', k', j') \\ &= L(n_0, k_0) L(n', k'), \end{aligned}$$

which is what we had to prove. ∎

Lemma 3.4. *The function*

$$M(n,k,j) = \binom{n}{j}\binom{k+j}{n}$$

is a ***TLP*** *function.*

Proof. Clearly, $M(n,k,j) = 0$, for $j > n$. In order to show that $M(n,k,j)$ is a **TLP** function, we therefore need to show that, for any prime p,

$$M(n_0 + n'p, k_0 + k'p, j_0 + j'p) \equiv M(n_0, k_0, j_0) M(n', k', j') \pmod{p}, \tag{16}$$

provided that $0 \leqslant n_0, k_0, j_0 < p$ and $n', k', j' \geqslant 0$. Observe that in the case $j_0 > n_0$ both sides of the congruence (16) vanish because of the Lucas congruences (8) for the binomial coefficients. We may therefore proceed under the assumption that $j_0 \leqslant n_0$.

Writing $[x^n]f(x)$ for the coefficient of x^n in the polynomial $f(x)$, we begin with the simple observation that

$$\binom{k+j}{n} = [x^n](1+x)^{k+j}.$$

Modulo p, we have

$$(1+x)^{k+j} = (1+x)^{k_0+j_0}(1+x)^{(k'+j')p} \equiv (1+x)^{k_0+j_0}(1+x^p)^{k'+j'} \pmod{p}.$$

Since $0 \leqslant k_0 + j_0 < 2p$, extracting the coefficient of $x^n = x^{n_0}(x^p)^{n'}$ from this product results in the congruence

$$\binom{k+j}{n} \equiv \binom{k_0+j_0}{n_0}\binom{k'+j'}{n'} + \binom{k_0+j_0}{n_0+p}\binom{k'+j'}{n'-1} \pmod{p}.$$

Note that, under our assumption that $j_0 \leqslant n_0$, the second term on the right-hand side of this congruence vanishes (since $n_0 + p \geqslant j_0 + p > j_0 + k_0$). This, along with (8), proves (16). $\square$

Corollary 3.5. *The function*

$$L(n,k) = \binom{n}{k}\binom{2k}{n}$$

is a ***DLP*** *function.*

Proof. Set $j = k$ in Lemma 3.4. $\square$

Lemma 3.6. *The function*

$$L(n,k) = 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}$$

is a ***DLP*** *function.*

Proof. Let p be a prime. As usual, we write $n = n_0 + n'p$ and $k = k_0 + k'p$ where $0 \leqslant n_0 < p$ and $0 \leqslant k_0 < p$. In light of (8) and (13), the simple observation

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2, \tag{17}$$

demonstrates that the sequence of central binomial coefficients is an **LP** function. We claim that

$$\frac{(3k)!}{k!^3} = \binom{3k}{k}\binom{2k}{k}$$

is an **LP** function as well. From the Lucas congruences for the central binomials, that is

$$\binom{2k}{k} \equiv \binom{2k_0}{k_0}\binom{2k'}{k'} \pmod{p},$$

we observe that $\binom{2k}{k}$ is divisible by p if $2k_0 \geqslant p$. Hence, we only need to show the congruences

$$\frac{(3k)!}{k!^3} \equiv \frac{(3k_0)!}{k_0!^3}\frac{(3k')!}{k'!^3} \pmod{p} \tag{18}$$

under the assumption that $k_0 < p/2$. Note that

$$\begin{aligned}
\binom{3k}{k} &= [x^k](1+x)^{3k} \\
&\equiv [x^{k_0}(x^p)^{k'}](1+x)^{3k_0}(1+x^p)^{3k'} \pmod{p} \\
&= \binom{3k_0}{k_0}\binom{3k'}{k'} + \binom{3k_0}{k_0+p}\binom{3k'}{k'-1} + \binom{3k_0}{k_0+2p}\binom{3k'}{k'-2}.
\end{aligned}$$

In the case $k_0 < p/2$, we have $k_0 + p > 3k_0$, so that the last two terms on the right-hand side vanish. This proves (18).

Next, we claim that

$$\binom{n}{3k}\frac{(3k)!}{k!^3} \equiv \binom{n_0}{3k_0}\frac{(3k_0)!}{k_0!^3}\binom{n'}{3k'}\frac{(3k')!}{k'!^3} \pmod{p}. \tag{19}$$

By congruence (18), both sides vanish modulo p if $3k_0 \geqslant p$. On the other hand, if $3k_0 < p$, then the usual argument shows that

$$\binom{n}{3k} \equiv [x^{3k_0}(x^p)^{3k'}](1+x)^{n_0}(1+x^p)^{n'} = \binom{n_0}{3k_0}\binom{n'}{3k'} \pmod{p}.$$

In combination with (18), this proves (19).

Finally, the congruences $L(n,k) \equiv L(n_0,k_0)L(n',k')$, that is

$$3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3} \equiv 3^{n_0-3k_0}\binom{n_0}{3k_0}\frac{(3k_0)!}{k_0!^3}3^{n'-3k'}\binom{n'}{3k'}\frac{(3k')!}{k'!^3} \pmod{p}, \tag{20}$$

follow from Fermat's little theorem and the fact that both sides vanish if $3k_0 > n_0$ or $3k' > n'$. $\square$

We are now in a comfortable position to prove Theorem 3.1 for all but two of the sporadic Apéry-like sequences. To show that sequences (η) and s_{18} satisfy Lucas congruences as well requires considerable additional effort, and the corresponding proofs are given in Section 4.

Proof of Theorem 3.1. Recall from (17) that the sequence of central binomial coefficients is an **LP** function. Further armed with (12) as well as Corollary 3.5 and Lemma 3.6, the claimed Lucas congruences for the sequences (a), (b), (c), (d), (f), (α), (ϵ), (γ), s_{10}, s_7 follow from (13). It remains to consider the sequences (g), (δ), (ζ) as well as (η) and s_{18}.

Sequence (g) can be written as

$$A_g(n) = \sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} F(k),$$

where $F(k) = \sum_{l=0}^{k} \binom{k}{l}^3$ are the Franel numbers (sequence (a)), which we already know to be an **LP** function. As a consequence of Fermat's little theorem, the sequence a^n is an **LP** function for any integer a. Hence, equation (13) applies to show that $A_g(n)$ is an **LP** function.

In order to see that sequence (δ) satisfies the Lucas congruences as well, it suffices to observe that $L(n,k) = \binom{n+k}{k}$ is almost a **DLP** function, that is, it satisfies the congruences (11) but does not vanish for $k > n$. This is enough to conclude from Lemma 3.6 that

$$L(n,k) = 3^{n-3k} \binom{n}{3k} \binom{n+k}{k} \frac{(3k)!}{k!^3}$$

is a **DLP** function. Since this is the summand of sequence (δ), the desired Lucas congruences again follow from (13).

On the other hand, for sequence (ζ), we observe that

$$L(n,k) = \sum_{j=0}^{n} \binom{n}{j} \binom{k}{j} \binom{k+j}{n}$$

satisfies the congruences (11) by Lemma 3.3 because the summand is a **TLP** function by Lemma 3.4. Hence, $\binom{n}{k}^2 L(n,k)$ is a **DLP** function. Writing sequence (ζ) as

$$A_\zeta(n) = \sum_{k=0}^{n} \binom{n}{k}^2 L(n,k),$$

the claimed congruences once more follow from (13). ☐

4 Proofs for the two remaining sequences

The proof of the Lucas congruences in the previous section does not readily extend to the sequences (η) and s_{18} from Table 2, because, in contrast to the

other cases, the known binomial sums for these sequences do not have the property that their summands satisfy the double Lucas property. Let us first note that the binomial sums for s_{18} and sequence (η), given in (7) and Table 2, can be simplified at the expense of working with binomial coefficients with negative entries. Namely, we have

$$s_{18}(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{2n-3k}{n} \tag{21}$$

and

$$A_\eta(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}, \tag{22}$$

where, as usual, for any integer $m \geqslant 0$ and any number x, we define

$$\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}.$$

For instance, the equivalence between (7) and (21) is a simple consequence of the fact that, for integers $n \geqslant 0$ and $l = n-k$,

$$(-1)^k \binom{2n-3k}{n} = (-1)^{k+n} \binom{-n+3k-1}{n} = (-1)^l \binom{2n-3l-1}{n}. \tag{23}$$

For the first equality, we used that, for integers $b \geqslant 0$,

$$\begin{aligned} \binom{a}{b} &= \frac{a(a-1)\cdots(a-b+1)}{b!} \\ &= (-1)^b \frac{(-a)(-a+1)\cdots(-a+b-1)}{b!} = (-1)^b \binom{-a+b-1}{b}. \end{aligned} \tag{24}$$

The following result generalizes the Lucas congruences for the sequence $s_{18}(n)$.

Theorem 4.1. *Suppose that $B(n,k)$ is a* ***DLP*** *function with the property that $B(n,k) = B(n, n-k)$. Then, the sequence*

$$A(n) = \sum_{k=0}^{n} (-1)^k B(n,k) \binom{2n-3k}{n}$$

is an ***LP*** *function, that is, $A(n)$ satisfy the Lucas congruences (9).*

Proof. Let p be a prime and let $n \geqslant 0$ be an integer. Write $n = n_0 + n'p$ and $k = k_0 + k'p$, where $0 \leqslant n_0 < p$ and $0 \leqslant k_0 < p$ and n', k' are nonnegative integers. We have to show that

$$A(n) \equiv A(n_0) A(n') \pmod{p}. \tag{25}$$

In the sequel, we denote

$$C(n,k) = (-1)^k B(n,k) \binom{2n-3k}{n}.$$

For $k_0 \leqslant n_0/3$, we have $2n_0 - 3k_0 \geqslant n_0 \geqslant 0$ and $2n_0 - 3k_0 \leqslant 2n_0 < n_0 + p$. Hence, by the usual argument, we have

$$\begin{aligned}\binom{2n-3k}{n} &\equiv [x^{n_0}(x^p)^{n'}](1+x)^{2n_0-3k_0}(1+x^p)^{2n'-3k'} \pmod p\\ &\equiv \binom{2n_0-3k_0}{n_0}\binom{2n'-3k'}{n'} \pmod p.\end{aligned}$$

Hence, we find that, when $k_0 \leqslant n_0/3$,

$$C(n,k) \equiv C(n_0,k_0)C(n',k') \pmod p. \tag{26}$$

For $n_0/3 < k_0 < 2n_0/3$, we have $n_0 > 2n_0 - 3k_0 > 0$. By the same argument as above, we find that

$$\binom{2n-3k}{n} \equiv 0 \pmod p, \tag{27}$$

and hence $C(n,k) \equiv C(n_0,k_0) \equiv 0$ modulo p.

Finally, consider the case $n_0 \geqslant 1$ and $2n_0/3 \leqslant k_0 \leqslant n_0$. In that case, $-p < -n_0 \leqslant 2n_0 - 3k_0 \leqslant 0$ or, equivalently, $0 < 2n_0 - 3k_0 + p \leqslant p$. Hence, we have, modulo p,

$$\begin{aligned}\binom{2n-3k}{n} &\equiv [x^{n_0}(x^p)^{n'}](1+x)^{2n_0-3k_0+p}(1+x^p)^{2n'-3k'-1}\\ &\equiv \binom{2n_0-3k_0+p}{n_0}\binom{2n'-3k'-1}{n'}\\ &\equiv \binom{2n_0-3k_0}{n_0}\binom{2n'-3k'-1}{n'},\end{aligned} \tag{28}$$

because, for any integers A, B and m such that $0 \leqslant m < p$,

$$\begin{aligned}\binom{A+Bp}{m} &= \frac{1}{m!}(A+Bp)(A+Bp-1)\cdots(A+Bp-m+1)\\ &\equiv \frac{1}{m!}A(A-1)\cdots(A-m+1) = \binom{A}{m} \pmod p.\end{aligned} \tag{29}$$

Set $l' = n' - k'$. Applying (23) to the second binomial factor in (28), we find that

$$\binom{2n-3k}{n} \equiv (-1)^{n'}\binom{2n_0-3k_0}{n_0}\binom{2n'-3l'}{n'} \pmod p.$$

In combination with the assumed symmetry of $B(n,k)$, we therefore have that, when $n_0 \geqslant 1$ and $2n_0/3 \leqslant k_0 \leqslant n_0$,

$$C(n,k) \equiv C(n_0,k_0)C(n',n'-k') \pmod p. \tag{30}$$

We are now ready to combine all cases. First, suppose that $n_0 \geqslant 1$. Noting that $k \leqslant n/3$ implies $k' \leqslant n'/3$, and using (26), (27) and (30), we conclude that, modulo p,

$$\begin{aligned}
A(n) &= \sum_{k_0=0}^{p-1} \sum_{k'=0}^{n'} C(n,k) \equiv \sum_{k_0=0}^{n_0} \sum_{k'=0}^{n'} C(n,k) \\
&\equiv \sum_{k_0=0}^{\lfloor n_0/3 \rfloor} \sum_{k'=0}^{n'} C(n,k) + \sum_{k_0=\lceil 2n_0/3 \rceil}^{n_0} \sum_{k'=0}^{n'} C(n,k) \\
&\equiv \sum_{k_0=0}^{\lfloor n_0/3 \rfloor} C(n_0,k_0) \sum_{k'=0}^{n'} C(n',k') + \sum_{k_0=\lceil 2n_0/3 \rceil}^{n_0} C(n_0,k_0) \sum_{k'=0}^{n'} C(n',n'-k') \\
&= \left[\sum_{k_0=0}^{\lfloor n_0/3 \rfloor} C(n_0,k_0) + \sum_{k_0=\lceil 2n_0/3 \rceil}^{n_0} C(n_0,k_0) \right] \sum_{k'=0}^{n'} C(n',k') \\
&= A(n_0)A(n'),
\end{aligned}$$

which is what we wanted to prove. The case $n_0 = 0$ is simpler, and we only have to use (26) to again conclude that (25) holds. $\square$

Corollary 4.2. *The sequence $s_{18}(n)$ satisfies the Lucas congruences (9).*

Proof. Recall from the discussion in Section 3 that

$$B(n,k) = \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$$

is a **DLP** function. Obviously, $B(n,k) = B(n,n-k)$. Hence, Theorem 4.1 applies to show that $s_{18}(n)$, in the form (21) satisfies the Lucas congruences (9). $\square$

Next, we prove that the sequence (η), which corresponds to the choice $a = 3$ in Theorem 4.3, satisfies Lucas congruences as well.

Theorem 4.3. *Let $a \in \{1,3\}$. Then, the sequence*

$$A(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \tag{31}$$

is an ***LP*** *function, that is, $A(n)$ satisfy the Lucas congruences (9).*

Proof. Let p be a prime and let $n \geqslant 0$ be an integer. As in the proof of Theorem 4.1, we write $n = n_0 + n'p$ and $k = k_0 + k'p$, where $0 \leqslant n_0 < p$ and $0 \leqslant k_0 < p$ and n', k' are nonnegative integers. Again, we have to show that

$$A(n) \equiv A(n_0)A(n') \pmod{p}. \tag{32}$$

Throughout the proof, let $d = \lfloor 3n_0/p \rfloor$.

If $k_0 \leqslant n_0/5$, then $4n_0 - 5k_0 \geqslant 3n_0 \geqslant 0$ and $4n_0 - 5k_0 \leqslant 4n_0 < 3n_0 + p$. Since $d = \lfloor 3n_0/p \rfloor$, we thus have $0 \leqslant 3n_0 - dp < p$ and $0 \leqslant 4n_0 - 5k_0 - dp < (3n_0 - dp) + p$. Therefore, modulo p,

$$\begin{aligned}\binom{4n-5k}{3n} &\equiv [x^{3n_0-dp}(x^p)^{3n'+d}](1+x)^{4n_0-5k_0-dp}(1+x^p)^{4n'-5k'+d} \\ &\equiv \binom{4n_0-5k_0-dp}{3n_0-dp}\binom{4n'-5k'+d}{3n'+d} \\ &\equiv \binom{4n_0-5k_0}{3n_0}\binom{4n'-5k'+d}{3n'+d},\end{aligned}$$

where in the last step we used that, modulo p,

$$\binom{4n_0-5k_0-dp}{3n_0-dp} = \binom{4n_0-5k_0-dp}{n_0-5k_0} \equiv \binom{4n_0-5k_0}{n_0-5k_0} = \binom{4n_0-5k_0}{3n_0}, \quad (33)$$

which follows from (29) because $0 \leqslant n_0 - 5k_0 < p$. In particular, we have

$$\begin{aligned}&\sum_{k_0=0}^{\lfloor n_0/5 \rfloor}\sum_{k'=0}^{n'}(-1)^k\binom{n}{k}^a\binom{4n-5k}{3n} \\ \equiv &\sum_{k_0=0}^{\lfloor n_0/5 \rfloor}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0}\sum_{k'=0}^{n'}(-1)^{k'}\binom{n'}{k'}^a\binom{4n'-5k'+d}{3n'+d}, \quad (34)\end{aligned}$$

and we observe that, for $d \in \{0, 1\}$,

$$A(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k+d}{3n+d}. \quad (35)$$

To see this, note that the the sum of the k-th and $(n-k)$-th term does not depend on the value of $d \in \{0, 1\}$. Indeed, using (24), Pascal's relation and (24) again, we deduce that

$$\begin{aligned}&\binom{4n-5k+1}{3n+1} + (-1)^n\binom{4n-5(n-k)+1}{3n+1} \\ = &\binom{4n-5k+1}{3n+1} - \binom{4n-5k-1}{3n+1} \\ = &\left[\binom{4n-5k+1}{3n+1} - \binom{4n-5k}{3n+1}\right] + \left[\binom{4n-5k}{3n+1} - \binom{4n-5k-1}{3n+1}\right] \\ = &\binom{4n-5k}{3n} + \binom{4n-5k-1}{3n} \\ = &\binom{4n-5k}{3n} + (-1)^n\binom{4n-5(n-k)}{3n}.\end{aligned}$$

Next, suppose that $n_0 \geqslant 1$ and $4n_0/5 \leqslant k_0 \leqslant n_0$. In that case, $-p < -n_0 \leqslant 4n_0 - 5k_0 \leqslant 0$ or, equivalently, $0 < 4n_0 - 5k_0 + p \leqslant p$. Hence, we have, modulo p,

$$\begin{aligned}\binom{4n-5k}{3n} &\equiv [x^{3n_0-dp}(x^p)^{3n'+d}](1+x)^{4n_0-5k_0+p}(1+x^p)^{4n'-5k'-1}\\ &\equiv \binom{4n_0-5k_0+p}{3n_0-dp}\binom{4n'-5k'-1}{3n'+d}.\end{aligned}$$

We rewrite the first binomial factor as follows, applying first (24) and then (29) twice, to find that, with $l_0 = n_0 - k_0$, modulo p,

$$\begin{aligned}\binom{4n_0-5k_0+p}{3n_0-dp} &= (-1)^{n_0+d}\binom{4n_0-5l_0-(d+1)p-1}{3n_0-dp}\\ &\equiv (-1)^{n_0+d}\binom{4n_0-5l_0-dp-1}{3n_0-dp}\\ &= (-1)^{n_0+d}\binom{4n_0-5l_0-dp-1}{n_0-5l_0-1}\\ &\equiv (-1)^{n_0+d}\binom{4n_0-5l_0-1}{n_0-5l_0-1}\\ &= (-1)^{n_0+d}\binom{4n_0-5l_0-1}{3n_0}.\end{aligned}$$

Here, we proceeded under the assumption that $n_0 - 5l_0 > 0$. It is straightforward to check that the final congruence also holds when $n_0 = 5l_0$, because then the binomial coefficients vanish modulo p. We conclude that, when $n_0 \geqslant 1$ and $4n_0/5 \leqslant k_0 \leqslant n_0$,

$$(-1)^k\binom{4n-5k}{3n} \equiv (-1)^{l_0}\binom{4n_0-5l_0-1}{3n_0}(-1)^{k'+d}\binom{4n'-5k'-1}{3n'+d} \pmod p.$$

In particular, we have

$$\begin{aligned}&\sum_{k_0=\lceil 4n_0/5\rceil}^{n_0}\sum_{k'=0}^{n'}(-1)^k\binom{n}{k}^a\binom{4n-5k}{3n}\\ &\equiv \sum_{k_0=\lceil 4n_0/5\rceil}^{n_0}(-1)^{l_0}\binom{n_0}{l_0}^a\binom{4n_0-5l_0-1}{3n_0}\sum_{k'=0}^{n'}(-1)^{k'+d}\binom{n'}{k'}^a\binom{4n'-5k'-1}{3n'+d}\\ &= \sum_{k_0=0}^{\lfloor n_0/5\rfloor}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0-1}{3n_0}\sum_{k'=0}^{n'}(-1)^{k'+d}\binom{n'}{k'}^a\binom{4n'-5k'-1}{3n'+d},\end{aligned} \tag{36}$$

and we observe that, for integers $d \geqslant 0$,

$$\sum_{k=0}^{n}(-1)^{k+d}\binom{n}{k}^a\binom{4n-5k-1}{3n+d} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k+d}{3n+d}$$

because, by (24),

$$(-1)^k \binom{4n-5k+d}{3n+d} = (-1)^{(n-k)+d} \binom{4n-5(n-k)-1}{3n+d}.$$

Therefore, we can combine (34) and (36) into

$$\begin{aligned}
&\sum_{\substack{k_0=0\\ k_0 \leqslant n_0/5 \text{ or } k_0 \geqslant 4n_0/5}}^{n_0} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \\
\equiv \quad & A(n_0) \sum_{k'=0}^{n'} (-1)^{k'} \binom{n'}{k'}^a \binom{4n'-5k'+d}{3n'+d} \pmod{p}, \qquad (37)
\end{aligned}$$

which holds for all $0 \leqslant n_0 < p$ (recall from the discussion at the beginning of this section that $A(n_0)$, like sequence (η), can be represented as in Table 2).

On the other hand, suppose that $n_0/5 < k_0 < 4n_0/5$. Set $f = \lfloor (4n_0 - 5k_0)/p \rfloor$. Since $0 < 4n_0 - 5k_0 < 3n_0 < 3p$, we have $f \in \{0, 1, 2\}$. The usual arguments show that, modulo p,

$$\begin{aligned}
\binom{4n-5k}{3n} \quad &\equiv \quad [x^{3n_0-dp}(x^p)^{3n'+d}](1+x)^{4n_0-5k_0-fp}(1+x^p)^{4n'-5k'+f} \\
&\equiv \quad \binom{4n_0-5k_0-fp}{3n_0-dp} \binom{4n'-5k'+f}{3n'+d} \\
&\equiv \quad \binom{4n_0-5k_0}{3n_0-dp} \binom{4n'-5k'+f}{3n'+d}. \qquad (38)
\end{aligned}$$

We are now in a position to begin piecing everything together. To do so, we consider individually the cases corresponding to the value of $d \in \{0, 1, 2\}$.

First, suppose $d = 0$ or $d = 1$. Congruence (37) coupled with (35) implies that

$$\sum_{\substack{k_0=0\\ k_0 \leqslant n_0/5 \text{ or } k_0 \geqslant 4n_0/5}}^{n_0} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \equiv A(n_0)A(n') \pmod{p}.$$

To conclude the desired congruence (32), it therefore only remains to show that

$$\sum_{k_0=\lfloor n_0/5 \rfloor+1}^{\lceil 4n_0/5 \rceil-1} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \equiv 0 \pmod{p}. \qquad (39)$$

This is easily seen in the case $d = 0$, because then each term of this sum vanishes modulo p. Equivalently, for $d = 0$, (38) vanishes whenever $n_0/5 < k_0 < 4n_0/5$ (because $0 \leqslant 4n_0 - 5k_0 - fp \leqslant 4n_0 - 5k_0 < 3n_0$). On the other hand, if $d = 1$, we claim that the sum (39) vanishes modulo p because the terms corresponding to (k_0, k') and $(k_0, n' - k')$ cancel each other. To see that, observe first that, for

$d=1$, (38) vanishes whenever $n_0/5 < k_0 < 4n_0/5$ and $f = \lfloor (4n_0 - 5k_0)/p \rfloor \neq 0$ (because $0 \leqslant 4n_0 - 5k_0 - fp \leqslant 4n_0 - 5k_0 - p < 3n_0 - p$ if $f \in \{1,2\}$). Therefore, for the term corresponding to (k_0, k'),

$$(-1)^k \binom{4n-5k}{3n} \equiv (-1)^{k_0} \binom{4n_0 - 5k_0}{3n_0 - p} (-1)^{k'} \binom{4n' - 5k'}{3n'+1} \pmod p,$$

while, for the term corresponding to $(k_0, n'-k')$ with $j = k_0 + (n'-k')p$,

$$\begin{aligned}
(-1)^j \binom{4n-5j}{3n} &\equiv (-1)^{k_0} \binom{4n_0 - 5k_0}{3n_0 - p} (-1)^{n'-k'} \binom{4n' - 5(n'-k')}{3n'+1} \\
&\equiv (-1)^{k_0} \binom{4n_0 - 5k_0}{3n_0 - p} (-1)^{k'+1} \binom{4n' - 5k'}{3n'+1} \\
&\equiv -(-1)^k \binom{4n-5k}{3n} \pmod p,
\end{aligned}$$

where we applied (24) for the second congruence. It is now immediate to see that the sum (39) indeed vanishes modulo p for $d=1$.

It remains to prove the Lucas congruences (32) in the case $d=2$. Using (37), we have

$$A(n) \equiv A(n_0) \sum_{k'=0}^{n'} (-1)^{k'} \binom{n'}{k'}^a \binom{4n' - 5k' + 2}{3n'+2} + M \pmod p,$$

where

$$M := \sum_{k_0 = \lfloor n_0/5 \rfloor + 1}^{\lceil 4n_0/5 \rceil - 1} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n}.$$

Combining this congruence with the identity

$$A(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a \left[\binom{4n-5k+2}{3n+2} - \binom{4n-5k}{3n+2} \right],$$

which can be deduced along the same lines as (35), we find that

$$A(n) \equiv A(n_0)A(n') + A(n_0) \sum_{k'=0}^{n'} (-1)^{k'} \binom{n'}{k'}^a \binom{4n' - 5k'}{3n'+2} + M \pmod p. \quad (40)$$

We have, by (38), modulo p,

$$\begin{aligned}
M &\equiv \sum_{k_0 = \lfloor n_0/5 \rfloor + 1}^{\lceil 4n_0/5 \rceil - 1} (-1)^{k_0} \binom{n_0}{k_0}^a \binom{4n_0 - 5k_0}{3n_0 - 2p} \sum_{k'=0}^{n'} (-1)^{k'} \binom{n'}{k'}^a \binom{4n' - 5k' + f}{3n'+2} \\
&\equiv \sum_{k_0 = \lfloor n_0/5 \rfloor + 1}^{\lceil 4n_0/5 \rceil - 1} (-1)^{k_0} \binom{n_0}{k_0}^a \binom{4n_0 - 5k_0}{3n_0 - 2p} \sum_{k'=0}^{n'} (-1)^{k'} \binom{n'}{k'}^a \binom{4n' - 5k'}{3n'+2},
\end{aligned}$$

where the last congruence is a consequence of the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k+1}{3n+2}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k}{3n+2}$$

(which follows from (24) and replacing k with $n-k$) and the fact that (38) vanishes for $n_0/5<k_0<4n_0/5$ if $f=2$. Using this value of M in (40), we find that the desired Lucas congruence (32) follows, if we can show that

$$A(n_0)+\sum_{k_0=\lfloor n_0/5\rfloor+1}^{\lceil 4n_0/5\rceil-1}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0-2p}\equiv 0\pmod p. \tag{41}$$

Note that, if $k_0\leqslant n_0/5$, then, by (29) and (33),

$$\binom{4n_0-5k_0}{3n_0-2p}\equiv\binom{4n_0-5k_0-2p}{3n_0-2p}\equiv\binom{4n_0-5k_0}{3n_0}\pmod p. \tag{42}$$

A similar argument, combined with (24), shows that the congruence (42) also holds if $k_0\geqslant 4n_0/5$. We therefore find that (41) is equivalent to

$$\sum_{k_0=0}^{n_0}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0-2p}\equiv 0\pmod p.$$

The next lemma proves that this congruence indeed holds provided that $a\in\{1,3\}$. □

Lemma 4.4. *Let p be a prime, and $a\in\{1,2,3\}$. Then we have, for all n such that $2p/3\leqslant n<p$,*

$$\sum_{k=0}^{n}(-1)^{ak}\binom{n}{k}^a\binom{4n-5k}{3n-2p}\equiv 0\pmod p.$$

Proof. To prove these congruences we employ N. Calkin's technique [Cal98] for proving similar divisibility results for sums of powers of binomials (58). Denoting $r=p-n$, we have, by (24) and (29),

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{ak}\binom{n}{k}^a\binom{4n-5k}{3n-2p}&=\sum_{k=0}^{p-r}(-1)^{ak}\binom{p-r}{k}^a\binom{4p-4r-5k}{p-3r}\\&=\sum_{k=0}^{p-r}\binom{k-p+r-1}{k}^a\binom{4p-4r-5k}{p-3r}\\&\equiv\sum_{k=0}^{p-r}\binom{k+r-1}{k}^a\binom{4p-4r-5k}{p-3r}\pmod p.\end{aligned}$$

Clearly,

$$\binom{k+r-1}{k} = \frac{(k+1)(k+2)\cdots(k+r-1)}{(r-1)!} = \frac{(k+1)_{r-1}}{(r-1)!}, \tag{43}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$ denotes the Pochhammer symbol (in particular, $(x)_0 = 1$). Likewise,

$$\binom{4p-4r-5k}{p-3r} = \frac{(3p-r-5k+1)_{p-3r}}{(p-3r)!}$$

Since $(r-1)!$ and $(p-3r)!$ are not divisible by p, we have to show that

$$\sum_{k=0}^{p-r} (k+1)_{r-1}^{a} (3p-r-5k+1)_{p-3r} \equiv 0 \pmod{p}. \tag{44}$$

Since the polynomials $(x)_k, (x)_{k-1}, \ldots, (x)_0$ form an integer basis for the space of all polynomials with integer coefficients and degree at most k, there exist integers $c_0, c_1, \ldots, c_N$ with $N = (a-1)(r-1) + p - 3r$ so that

$$(k+1)_{r-1}^{a-1} (3p-r-5k+1)_{p-3r} = \sum_{j=0}^{N} c_j (k+r)_j.$$

Then the left-hand side of (44) becomes

$$\begin{aligned} \sum_{k=0}^{p-r} (k+1)_{r-1} \sum_{j=0}^{N} c_j (k+r)_j &= \sum_{j=0}^{N} c_j \sum_{k=0}^{p-r} (k+1)_{r-1} (k+r)_j \\ &= \sum_{j=0}^{N} c_j \sum_{k=0}^{p-r} (k+1)_{r+j-1} \\ &= \sum_{j=0}^{N} c_j \frac{(p-r+1)_{r+j}}{r+j}, \end{aligned} \tag{45}$$

where we used

$$(x)_k - (x-1)_k = k(x)_{k-1}$$

to evaluate

$$\sum_{k=0}^{p-r} (k+1)_{r+j-1} = \sum_{k=0}^{p-r} \frac{(k+1)_{r+j} - (k)_{r+j}}{r+j} = \frac{(p-r+1)_{r+j}}{r+j}.$$

The desired congruence therefore follows if we can show that

$$\frac{(p-r+1)_{r+j}}{r+j} \equiv 0 \pmod{p} \tag{46}$$

for all $j = 0, 1, \ldots, N$. Since $r > 0$ and $j \geqslant 0$, the numerator $(p - r + 1)_{r+j}$ is always divisible by p. The congruences (46) thus follow if $r + j < p$ for all j, or, equivalently, $r + N < p$. Since

$$r + N = (a - 1)(r - 1) + p - 2r,$$

we have $r + N < p$ if and only if

$$(a - 1)(r - 1) < 2r.$$

Clearly, this inequality holds for all $r \geqslant 1$ if and only if $a \leqslant 3$. □

Remark 4.5. Numerical evidence suggests that the values $a \in \{1, 3\}$ in Theorem 4.3 are the only choices for which the sequence (31) satisfies Lucas congruences. In light of Lemma 4.4, it is natural to ask if there are additional values of a and ε, for which the sequence

$$\sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k}^a \binom{4n - 5k}{3n}$$

satisfies Lucas congruences. Empirically, this does not appear to be the case. In particular, for $a = 2$ this sequence does not satisfy Lucas congruences for either $\varepsilon = 0$ or $\varepsilon = 1$.

5 Periodicity of residues

The Apéry numbers satisfy

$$A(n) \equiv (-1)^n \pmod 3, \tag{47}$$

and so are periodic modulo 3. As in the case of the congruences (4), which show that the Apéry numbers are also periodic modulo 8, the congruences (47) were first conjectured in [CCC80] and then proven in [Ges82]. We say that a sequence $C(n)$ is *eventually periodic* if there exists an integer $M > 0$ such that $C(n + M) = C(n)$ for all sufficiently large n. An initial numerical search suggests that each sporadic Apéry-like sequence listed in Tables 1 and 2 can only be eventually periodic modulo a prime p if $p \leqslant 5$. As an application of Theorem 3.1, we prove this claim next.

Corollary 5.1. *None of the sequences from Tables 1 and 2 is eventually periodic modulo p for any prime $p > 5$.*

Proof. Gessel [Ges82] shows that, if a sequence $C(n)$ satisfies the Lucas congruences (9) modulo p and is eventually periodic modulo p, then $C(n) \equiv C(1)^n$ modulo p for all $n = 0, 1, \ldots, p - 1$.

For instance, let $C(n)$ be the Almkvist–Zudilin sequence (δ). Then, $C(1) = 3$, $C(2) = 9$ and $C(3) = 3$. Suppose $C(n)$ was eventually periodic modulo p. Then p has to divide $C(3) - C(1)^3 = -24$, which implies that $p \in \{2, 3\}$.

In Table 3 we list, for each sequence, the primes dividing both $C(2)-C(1)^2$ and $C(3)-C(1)^3$. The fact, that all these primes are at most 5, proves our claim. $\square$

(a)	(b)	(c)	(d)	(f)	(g)	(δ)	(η)	(α)	(ϵ)	(ζ)	(γ)	(s_7)	(s_{10})	(s_{18})
$2,3$	$2,5$	$2,3$	2	$2,3$	$2,3$	$2,3$	$2,5$	$2,3$	$2,3$	$2,3$	$2,3$	2	2	$2,3$

Table 3: The primes dividing both $C(2)-C(1)^2$ and $C(3)-C(1)^3$, for each sequence $C(n)$ from Tables 1 and 2.

In fact, as another simple consequence of Theorem 3.1, we observe that the Apéry-like sequences are in fact eventually periodic modulo each of the primes listed in Table 3.

Corollary 5.2. *Let $C(n)$ be any sequence from Tables 1 and 2.*

- $C(n) \equiv C(1) \pmod 2$ *for all* $n \geqslant 1$.
- $C(n) \equiv C(1) \pmod 3$ *for all* $n \geqslant 1$ *if* $C(n)$ *is one of* (c), (f), (g), (δ), (α), (ϵ), (ζ), s_{18}, *and* $C(n) \equiv (-1)^n \pmod 3$ *for all* $n \geqslant 0$ *if* $C(n)$ *is* (a) *or* (γ).
- $C(n) \equiv 3^n \pmod 5$ *for all* $n \geqslant 0$ *if* $C(n)$ *is* (b), *and* $C(n) \equiv 0 \pmod 5$ *for all* $n \geqslant 1$ *if* $C(n)$ *is* (η).

Proof. One can check that Table 3 does not change if we include only those primes p such that $C(n)-C(1)^n$ is divisible by p for all $n \in \{0,1,2,3,4\}$. For $n=0$, this is trivial since $C(0)=1$. Therefore, in each of the cases considered here, we have

$$C(n) \equiv C(1)^n \pmod p$$

for all $n \in \{0,1,\ldots,p-1\}$. For any $n \geqslant 0$, let $n = n_0 + n_1 p + \cdots + n_r p^r$ be the p-adic expansion of n. Then, by Theorem 3.1, we have

$$\begin{aligned} C(n) &\equiv C(n_0)C(n_1)\cdots C(n_r) \pmod p \\ &\equiv C(1)^{n_0+n_1+\cdots+n_r} \pmod p \\ &\equiv C(1)^n \pmod p. \end{aligned}$$

For the final congruence we used Fermat's little theorem. All claimed congruences then follow from the specific initial values of $C(n)$ modulo p. $\square$

More interestingly, the congruences (4) show that the Apéry numbers (sequence (γ)) are periodic modulo 8. We offer the following corresponding result for the Almkvist–Zudilin sequence (δ).

Theorem 5.3. *The Almkvist–Zudilin numbers*

$$Z(n) = \sum_{k=0}^{n} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

satisfy the congruences

$$Z(n) \equiv \begin{cases} 1, & \textit{if } n \textit{ is even,} \\ 3, & \textit{if } n \textit{ is odd,} \end{cases} \pmod 8. \tag{48}$$

Proof. It is shown in [Str14] that the numbers $(-1)^n Z(n)$ are the diagonal Taylor coefficients of the multivariate rational function

$$F(x_1, x_2, x_3, x_4) = \frac{1}{1 - (x_1 + x_2 + x_3 + x_4) + 27x_1x_2x_3x_4}. \tag{49}$$

That is, if

$$F(x_1, x_2, x_3, x_4) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} C(n_1, n_2, n_3, n_4) x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$$

is the Taylor expansion of the rational function F, then $Z(n) = (-1)^n C(n, n, n, n)$.

Given such a rational function as well as a reasonably small prime power p^r, Rowland and Yassawi [RY13] give an explicit algorithm for computing a finite state automaton, which produces the values of the diagonal coefficients modulo p^r. In the present case, this finite state automaton for the values $(-1)^n Z(n)$ modulo 8 turns out to be the same automaton as the one for the Apéry numbers modulo 8. Hence, the congruences (48) follow from the congruences (4). We refer to [RY13] for details on finite state automata and the algorithm to construct them from a multivariate rational generating function. □

Empirically, Theorem 5.3 is the only other interesting set of congruences, apart from the congruences (4), which demonstrates that an Apéry-like sequence is periodic modulo a prime power. More precisely, numerical evidence suggests that none of the sequences in Tables 1 and 2 is eventually periodic modulo p^r, for some $r > 1$, unless $p = 2$. Moreover, the only other instances modulo a power of 2 appear to be the following, less interesting, ones: sequences (d) and (α) are eventually periodic modulo 4 because all their terms, except the first, are divisible by 4; likewise, sequences (ε) and s_7 are eventually periodic modulo 8 because all their terms, except the first, are divisible by 8. We do not attempt to prove these claims here. We remark, however, that these claims can be established by the approach used in the proof of Theorem 5.3, provided that one is able to determine a computationally accessible analog of (49) for the sequence at hand.

6 Primes not dividing Apéry-like numbers

Using the Lucas congruences proved in Theorem 3.1, it is straightforward to verify whether or not a given prime divides some Apéry-like number.

Example 6.1. The values of Apéry numbers $A(0), A(1), \ldots, A(6)$ modulo 7 are $1, 5, 3, 3, 3, 5, 1$. Since 7 does not divide $A(0), A(1), \ldots, A(6)$, it follows from the Lucas congruences (9) that 7 does not divide any Apéry number. ◇

Arguing as in Example 6.1, one finds that the primes $2, 3, 7, 13, 23, 29, 43, 47, \ldots$ do not divide any Apéry number $A(n)$. E. Rowland and R. Yassawi [RY13] pose the question whether there are infinitely many such primes. Table 4 records, for each sporadic Apéry-like sequence, the primes below 100 which do not divide any of its terms, and the last column gives the proportion of primes below 10^4 with this property. Each Apéry-like sequence is specified by its label from [AvSZ11], which is also used in Tables 1 and 2. The alert reader will notice that Cooper's sporadic sequences (the ones with $d \neq 0$ in Table 2) are missing from Table 4. That is because these sequences turn out to be divisible by all primes. A more precise result for these sequences is proved at the end of this section.

(a)	3, 11, 17, 19, 43, 83, 89, 97	0.2994
(b)	2, 5, 13, 17, 29, 37, 41, 61, 73, 89	0.2897
(c)	2, 7, 13, 37, 61, 73	0.2962
(d)	3, 11, 17, 19, 43, 59, 73, 83, 89	0.2815
(f)	2, 5, 13, 17, 29, 37, 41, 61, 73, 97	0.2994
(g)	5, 11, 29, 31, 59, 79	0.2929
(δ)	2, 5, 7, 11, 13, 19, 29, 41, 47, 61, 67, 71, 73, 89, 97	0.6192
(η)	2, 3, 17, 19, 23, 31, 47, 53, 61	0.2897
(α)	3, 5, 13, 17, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 83, 89	0.5989
(ϵ)	3, 7, 13, 19, 23, 29, 31, 37, 43, 47, 61, 67, 73, 83, 89	0.6037
(ζ)	2, 5, 7, 13, 17, 19, 29, 37, 43, 47, 59, 61, 67, 71, 83, 89	0.6046
(γ)	2, 3, 7, 13, 23, 29, 43, 47, 53, 67, 71, 79, 83, 89	0.6168

Table 4: The primes below 100 not dividing Apéry-like numbers (sequence indicated in first column using the labels from [AvSZ11]) as well as the proportion of primes (in the last column) below $10,000$ not dividing any term

Example 6.1 shows that the first 7 values of the Apéry numbers modulo 7 are palindromic. Our next result, which was noticed by E. Rowland, shows that this is true for all primes.

Lemma 6.2. *For any prime p, and integers n such that $0 \leqslant n < p$, the Apéry numbers $A(n)$ satisfy the congruence*

$$A(n) \equiv A(p-1-n) \pmod{p}. \tag{50}$$

Proof. For n such that $0 \leqslant n < p$, we employ (24) and (29) to arrive at

$$\begin{aligned} A(p-1-n) &= \sum_{k=0}^{p-1} \binom{p-1-n}{k}^2 \binom{p-1-n+k}{k}^2 \\ &\equiv \sum_{k=0}^{p-1} \binom{n+k}{k}^2 \binom{n}{k}^2 = A(n) \pmod{p}, \end{aligned}$$

as claimed. ∎

Theorem 3.1 and Lemma 6.2, considered together, suggest that $e^{-1/2} \approx 60.65\%$ of the primes do not divide any Apéry number. Indeed, let us make the empirical assumption that the values $A(n)$ modulo p, for $n = 0, 1, \ldots, (p-1)/2$, are independent and uniformly random. Since one of the values $A(n)$ is congruent to 0 modulo p with probability $1/p$, it follows that the probability that p does not divide any of the $(p+1)/2$ first values is

$$\left(1 - \frac{1}{p}\right)^{(p+1)/2}. \tag{51}$$

By the Lucas congruences, shown in Theorem 3.1, and Lemma 6.2, p does not divide any of the $(p+1)/2$ first values if and only if p does not divide any Apéry number. In the limit $p \to \infty$, the proportion (51) becomes $e^{-1/2}$. Observe that this empirical prediction matches the numerical data in Table 4 rather well. We therefore arrive at the following conjecture.

Conjecture 6.3. *The proportion of primes not dividing any Apéry number $A(n)$ is $e^{-1/2}$.*

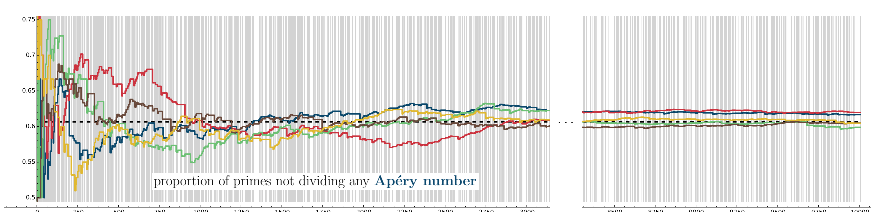

Figure 1: Proportion of primes (up to $10,000$) not dividing the sequences (δ), (α), (ϵ), (ζ), (γ), with the dotted line indicating $e^{-1/2}$. The Apéry sequence is plotted in blue. (We thank Arian Daneshvar for producing this plot.)

While Lemma 6.2 does not hold for the other Apéry-like numbers $C(n)$ from Tables 1 and 2, we make the weaker observation that if a prime $p > 5$ divides $C(n)$, where $0 \leqslant n < p$, then p also divides $C(p-1-n)$. We expect that this empirical observation can be proven in the spirit of the proof of Lemma 6.2, but do not pursue this theme further. We only note that it allows us to extend the heuristic leading to Conjecture 6.3 to the Apéry-like sequences (δ), (α), (ϵ), (ζ) from Table 2. In other words, we conjecture that, for each of these sequences, the proportion of primes not dividing any of the terms is again $e^{-1/2}$. Figure 1 visualizes some numerical evidence for this conjecture. On the other hand, for sequence (η) as well as the sequences from Table 1, the proportion of primes not dividing any of their terms appears to be about half of that, that is $e^{-1/2}/2 \approx 30.33\%$.

To explain this extra factor of $1/2$, we note that, for the Apéry-like numbers

$$A_b(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{n}, \tag{52}$$

Stienstra and Beukers [SB85] proved that, modulo p,

$$A_b\left(\frac{p-1}{2}\right) \equiv \begin{cases} 4a^2-2p, & \text{if } p=a^2+b^2,\ a \text{ odd},\\ 0, & \text{if } p\equiv 3 \pmod 4 \end{cases} \tag{53}$$

(and conjectured that the congruence should hold modulo p^2, which was later proved by Ahlgren and Ono [AO00]). In particular, congruence (53) makes it explicit that every prime $p \equiv 3 \pmod 4$ divides a term of this Apéry-like sequence. Note that, by a classical congruence of Gauss, the congruences (53) are equivalent, modulo p, to the congruences

$$A_b\left(\left\lfloor\frac{p}{2}\right\rfloor\right) \equiv \begin{cases} \binom{\lfloor p/2\rfloor}{\lfloor p/4\rfloor}^2, & \text{if } p\equiv 1 \pmod 4,\\ 0, & \text{otherwise,} \end{cases} \tag{54}$$

which are valid for any prime $p \neq 2$. The more general result in [SB85] also includes the cases A_a and A_c. Similar divisibility results appear to hold for the other Apéry-like numbers from Table 1, and it would be interesting to make these explicit.

On the other hand, the extra factor of $1/2$ in case of sequence (η) is explained by the following congruences, which resemble (54) remarkably well.

Theorem 6.4. *For any prime $p \neq 3$, we have that, modulo p,*

$$A_\eta\left(\left\lfloor\frac{p}{3}\right\rfloor\right) \equiv \begin{cases} (-1)^{\lfloor p/5\rfloor}\binom{\lfloor p/3\rfloor}{\lfloor p/15\rfloor}^3, & \textit{if } p\equiv 1,2,4,8 \pmod{15},\\ 0, & \textit{otherwise.} \end{cases} \tag{55}$$

Proof. Suppose that $p \equiv 2 \pmod 3$, and write $p = 3n+2$. The congruence (55) can be checked directly for $p = 2$ and $p = 5$, and so we may assume $p > 5$ in the sequel. Applying (43) to the definition of sequence (η) in Table 2, we have

$$\begin{aligned} A_\eta(n) &= \sum_{k=0}^{\lfloor n/5\rfloor} (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right)\\ &= \sum_{k=0}^{\lfloor n/5\rfloor} (-1)^k \binom{n}{k}^3 \left(\frac{(n-5k)_{3n}}{(3n)!} + \frac{(n-5k+1)_{3n}}{(3n)!}\right). \end{aligned} \tag{56}$$

Since $3n = p-2$ and $0 \leqslant k \leqslant n/5$, the term

$$\frac{(n-5k)_{3n}}{(3n)!} \tag{57}$$

is always divisible by p, unless $n-5k \in \{1,2\}$ (because, otherwise, one of the $p-2$ factors of $(n-5k)_{3n}$ is divisible by p, while $(3n)!$ is not). Note that $n-5k=1$ and $n-5k=2$ are equivalent to $k=(p-5)/15$ and $k=(p-8)/15$, respectively. However, $(p-5)/15$ cannot be an integer (since $p \neq 5$). We thus find that (57) vanishes modulo p unless $p \equiv 8 \pmod{15}$ and $k = \lfloor p/15\rfloor$, in

which case (57) is congruent to -1 modulo p. Combined with the analogous discussion for the corresponding second term in (56), we conclude that

$$\frac{(n-5k)_{3n}}{(3n)!} + \frac{(n-5k+1)_{3n}}{(3n)!} \equiv \begin{cases} 1, & \text{if } k = \lfloor p/15 \rfloor \text{ and } p \equiv 2 \pmod{15}, \\ -1, & \text{if } k = \lfloor p/15 \rfloor \text{ and } p \equiv 8 \pmod{15}, \\ 0, & \text{otherwise.} \end{cases}$$

Applying this to the sum (56) and combining the signs properly, we arrive at the congruences (55) when $p \equiv 2 \pmod 3$.

The case $p \equiv 1 \pmod 3$ is similar and a little bit simpler. □

In summary, we conjecture that the proportion of primes not dividing any term of the Apéry-like sequences in Tables 1 and 2 is as follows.

Conjecture 6.5.

- *Let $C(n)$ be one of the sequences of Table 1 or sequence (η). Then the proportion of primes not dividing any $C(n)$ is $\frac{1}{2}e^{-1/2}$.*
- *Let $C(n)$ be one of the sequences (δ), (α), (ϵ), (ζ), (γ) from Table 2. Then the proportion of primes not dividing any $C(n)$ is $e^{-1/2}$.*

In stark contrast, Cooper's sporadic sequences s_7, s_{10}, s_{18} from Table 2 are divisible by all primes. Indeed, let $C(n)$ denote any of these three sequences. Then,

$$C(p-1) \equiv 0 \pmod p$$

for all primes p. In fact, we can prove much more. For any given prime p, the last quarter (or third) of the first p terms of these sequences are divisible by p. In the case of sequence s_{10}, the sum of fourth powers of binomial coefficients, this is proved by N. Calkin [Cal98]. Indeed, among other divisibility results on sums of powers of binomials, Calkin shows that, for all integers $a \geqslant 0$, the sums

$$\sum_{k=0}^{n} \binom{n}{k}^{2a} \tag{58}$$

are divisible by all primes p in the range

$$n < p < n + 1 + \frac{n}{2a-1}.$$

In particular, in the case $a = 2$, we conclude that $s_{10}(n)$ is divisible by all primes p that satisfy $n < p < \frac{4n}{3} + 1$. Equivalently, we have

$$s_{10}(p-j) \equiv 0 \pmod p$$

whenever $1 \leqslant j \leqslant (p+2)/4$. Our final result proves the same phenomenon for Cooper's sporadic sequences s_7, s_{18}. We note that in each case, empirically, the bounds on j cannot be improved (with the expection of the case $p = 3$ for s_{18}; see Remark 6.7).

Theorem 6.6. *For any prime p, we have*

$$s_7(p-j) \equiv 0 \pmod p$$

whenever $1 \leqslant j \leqslant (p+1)/3$, and

$$s_{18}(p-j) \equiv 0 \pmod p$$

whenever $1 \leqslant j \leqslant (p+2)/4$.

Proof. For the sequence s_7, we want to show

$$\sum_{k=0}^{p-j} \binom{p-j}{k}^2 \binom{p-j+k}{k} \binom{2k}{p-j} \equiv 0 \pmod p,$$

for $1 \leqslant j \leqslant (p+1)/3$. Note that for $2k < p-j$ or $k > p-j$ the summand is already zero. Therefore, we assume that $p-j \geqslant k \geqslant (p-j)/2$. We write the summand as

$$\binom{p-j}{k}^2 \binom{p-j+k}{k} \binom{2k}{p-j} = \frac{(p-j+k)!(2k)!}{k!^3(p-j-k)!^2(2k-p+j)!},$$

and observe that the denominator is not divisible by p if $j \geqslant 1$. On the other hand, the factorial $(p-j+k)!$ in the numerator is divisible by p since

$$p-j+k \geqslant p-j+\left\lceil \frac{p-j}{2} \right\rceil \geqslant p,$$

where we used $j \leqslant (p+1)/3$ to verify the final inequality. Thus, for $1 \leqslant j \leqslant (p+1)/3$, the congruences $s_7(p-j) \equiv 0$ hold modulo p, as claimed.

We proceed similarly for $s_{18}(p-j)$, which is given by

$$\sum_{k=0}^{\lfloor (p-j)/3 \rfloor} (-1)^k \binom{p-j}{k} \binom{2k}{k} \binom{2(p-j-k)}{p-j-k} \left\{ \binom{2(p-j)-3k-1}{p-j} + \binom{2(p-j)-3k}{p-j} \right\},$$

and, using (43), write the summand as

$$\frac{(-1)^k(2k)!(2(p-j-k))!}{k!^3(p-j-k)!^3}(p-j-3k+1)_{p-j-1}(3p-3j-6k). \tag{59}$$

None of the terms in the denominator is divisible by p since $j \geqslant 1$. On the other hand, $(2(p-j-k))!$ in the numerator is divisible by p since

$$2(p-j-k) \geqslant 2\left(p-j-\left\lfloor \frac{p-j}{3} \right\rfloor\right) \geqslant p,$$

where we used $j \leqslant (p+2)/4$ for the final inequality. Therefore, for $1 \leqslant j \leqslant (p+2)/4$, each of the terms in the sum for $s_{18}(p-j)$ is a multiple of p, and we obtain the desired congruences. □

Remark 6.7. Employing (59), we observe that $s_{18}(n) \equiv 0 \pmod 3$ for $n \geqslant 1$, which reaffirms Corollary 5.2 for this sequence.

Finally, as noted in [Coo12], each of the sequences in Table 1 times $\binom{2n}{n}$ is an integer solution of (6) with $d \neq 0$. Observe that $\binom{2n}{n}$ is divisible by a prime p for all n such that $n < p \leqslant 2n$. This results in a (weaker) analog of Theorem 6.6 for these Apéry-like sequences, and implies, in particular, that these sequences are again divisible by all prime numbers.

7 Conclusion and open questions

In Sections 3 and 4, we showed that all sporadic solutions of (5) and (6), given in Tables 1 and 2, uniformly satisfy Lucas congruences. However, for two of these sequences, especially sequence (η), we had to resort to a rather technical analysis. We therefore wonder if there is a representation of these sequences from which the Lucas congruences can be deduced more naturally, based on, for instance the approaches of [SvS09] and [MV13], or [RY13]. More generally, it would be desirable to have a uniform approach to these congruences, possibly directly from the shape of the defining recurrences and associated differential equations. In another direction, it would be interesting to show that, as numerical evidence suggests, *all* of the Apéry-like sequences in fact satisfy the Dwork congruences (3).

The congruences (4) show that the Apéry numbers are periodic modulo 8, alternating between the values 1 and 5. As a consequence, the other residue classes $0, 2, 3, 4, 6, 7$ modulo 8 are never attained. On the other hand, the observations in Section 6 show that certain primes do not divide any Apéry number. This can be rephrased as saying that the residue class 0 is not attained by the Apéry numbers modulo these primes. This leads us to the question of which residue classes are not attained by Apéry-like numbers modulo prime powers p^α. In particular, are there interesting cases which are not explained by Sections 5 and 6?

The second part of congruence (53) makes it explicit that every prime $p \equiv 3 \pmod 4$ divides a term of the Apéry-like sequence (52). Is there a similarly explicit result which demonstrates that the Apéry numbers are divisible by infinitely many distinct primes?

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References

[AO00] S. Ahlgren and K. Ono. A Gaussian hypergeometric series evaluation and Apéry number congruences. *Journal für die reine und angewandte Mathematik*, 2000(518), January 2000.

[AvSZ11] G. Almkvist, D. van Straten, and W. Zudilin. Generalizations of Clausen's formula and algebraic transformations of Calabi–Yau differential equations. *Proc. Edinburgh Math. Soc.*, 54(2):273–295, 2011.

[AZ06] G. Almkvist and W. Zudilin. Differential equations, mirror maps and zeta values. In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 481–515. Amer. Math. Soc., Providence, RI, 2006.

[Apé79] R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

[Beu85] F. Beukers. Some congruences for the Apéry numbers. *Journal of Number Theory*, 21(2):141–155, October 1985.

[Beu87] F. Beukers. Another congruence for the Apéry numbers. *Journal of Number Theory*, 25(2):201–210, February 1987.

[Beu02] F. Beukers. On Dwork's accessory parameter problem. *Mathematische Zeitschrift*, 241(2):425–444, October 2002.

[Cal98] N. Calkin. Factors of sums of powers of binomial coefficients. *Acta Arithmetica*, 86:17–26, 1998.

[CCS10] H. H. Chan, S. Cooper, and F. Sica. Congruences satisfied by Apéry-like numbers. *International Journal of Number Theory*, 06(01):89, 2010.

[CCC80] S. Chowla, J. Cowles, and M. Cowles. Congruence properties of Apéry numbers. *Journal of Number Theory*, 12(2):188–190, May 1980.

[Coo12] S. Cooper. Sporadic sequences, modular forms and new series for $1/\pi$. *The Ramanujan Journal*, 29(1–3):163–183, 2012.

[Cos88] M. J. Coster. *Supercongruences*. PhD thesis, Universiteit Leiden, 1988.

[DS06] E. Deutsch and B. E. Sagan. Congruences for Catalan and Motzkin numbers and related sequences. *Journal of Number Theory*, 117(1):191–215, March 2006.

[Ego84] G. P. Egorychev. *Integral representation and the computation of combinatorial sums*, volume 59 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1984. Translated from the Russian by H. H. McFadden, Translation edited by Lev J. Leifman.

[Ges82] I. M. Gessel. Some congruences for Apéry numbers. *Journal of Number Theory*, 14(3):362–368, June 1982.

[KM15] C. Krattenthaler and T. W. Müller. Generalised Apéry numbers modulo 9. *Journal of Number Theory*, 147:708–720, February 2015.

[Luc78] E. Lucas. Sur les congruences des nombres Eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier. *Bull. Soc. Math. France*, 6:49–54, 1878.

[McI92] R. J. McIntosh. A generalization of a congruential property of Lucas. *Amer. Math. Monthly*, 99(3):231, March 1992.

[MV13] A. Mellit and M. Vlasenko. Dwork's congruences for the constant terms of powers of a Laurent polynomial. *Preprint*, June 2013. Available at: `http://arxiv.org/abs/1306.5811`.

[OS11] R. Osburn and B. Sahu. Supercongruences for Apéry-like numbers. *Advances in Applied Mathematics*, 47(3):631–638, September 2011.

[OS13] R. Osburn and B. Sahu. A supercongruence for generalized Domb numbers. *Functiones et Approximatio Commentarii Mathematici*, 48(1):29–36, March 2013.

[OSS14] R. Osburn, B. Sahu, and A. Straub. Supercongruences for sporadic sequences. *Proc. Edinburgh Math. Soc.*, 2014. To appear. arXiv:1312.2195.

[Poo79] A. v. d. Poorten. A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer*, 1(4):195–203, 1979.

[RY13] E. Rowland and R. Yassawi. Automatic congruences for diagonals of rational functions. *Preprint*, October 2013. arXiv:1310.8635.

[SvS09] K. Samol and D. van Straten. Dwork congruences and reflexive polytopes. *Preprint*, November 2009. Available at: `http://arxiv.org/abs/0911.0797`.

[SB85] J. Stienstra and F. Beukers. On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces. *Mathematische Annalen*, 271(2):269–304, April 1985.

[Str14] A. Straub. Multivariate Apéry numbers and supercongruences of rational functions. *Algebra & Number Theory*, 8(8):1985–2008, 2014.

[Zag09] D. Zagier. Integral solutions of Apery-like recurrences. In J. Harnad and P. Winternitz, editors, *Groups and symmetries. From Neolithic to John McKay*, volume 47. American Mathematical Society, 2009.